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1. Definition of function : Domain, mapping rule, co-domain and range.

A function : $x \overset{f}{\longmapsto} f(x)$

①. x: points that where the function f is defined, called domain. $D(f)$

② f: mapping rule, define the value of this function f at x, write as $f(x)$.

③. $f(x)$: If we collect all values of f, we would get a new set:

$R(f) = \{f(x) \mid x \in D(f)\}$, the range of $f(x)$

so we can see $R(f)$ is determinated by the $D(f)$ and mapping rule f.

Remark : the difference between co-domain and range:

$A \overset{f}{\longmapsto} B. \Rightarrow$

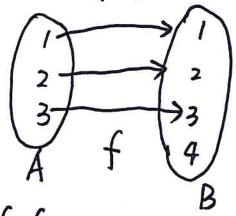


fig-1

$A = D(f)$: domain of f

always belongs to the max-domain of f.

B = co-domain : A set that always larger than the range of f, of course can be equal.

Ex. 1. In fig-1, $A = \{1, 2, 3\}$, so $D(f) = A = \{1, 2, 3\}$

And we define $\begin{cases} f(1) = 1 \\ f(2) = 2 \\ f(3) = 3 \end{cases} \Rightarrow R(f) = \{f(x) \mid x \in D(f)\} = \{1, 2, 3\}$

while our co-domain $B = \{1, 2, 3, 4\}$ so $R(f) \subsetneqq B$.

Ex. 2 Give the max-domain and range of following functions.

$$f(x) = \frac{1}{\sqrt{x^2-4}}$$

$D(f) = \{x \mid \sqrt{x^2-4} \neq 0\} \cap \{x \mid x^2-4 \geq 0\}$

$= \{x \mid x \neq \pm 2\} \cap \{x \mid x \geq 2 \text{ or } x \leq -2\}$

$= \{x \mid x > 2 \text{ or } x < -2\}$.

$R(f) = (0, +\infty)$

why? For we choose any $t \in (0, +\infty)$

let $f(x) = \frac{1}{\sqrt{x^2-4}} = t \Rightarrow x^2 = 4 + \frac{1}{t^2} \Rightarrow x_0 = \pm\sqrt{4+\frac{1}{t^2}} \in D($

which means we can find some x_0 make $f(x_0) = t$

this is the definition of range.

$$f(x) = \frac{1}{\cos x + \sin x}$$

$$D(f) = \{x \mid \cos x + \sin x \neq 0\}$$
$$= \{x \mid \sqrt{2}\sin(x+\tfrac{\pi}{4}) \neq 0\}$$
$$= \{x \mid x \neq k\pi - \tfrac{\pi}{4},\ k \in \mathbb{Z}\}.$$

for $\cos x + \sin x = \sqrt{2}\sin(x+\frac{\pi}{4}) \in [-\sqrt{2}, \sqrt{2}]$

and $\cos x + \sin x \neq 0$. so:

$$f(x) = \frac{1}{\cos x + \sin x} \in (-\infty, \tfrac{-\sqrt{2}}{2}] \cup [\tfrac{\sqrt{2}}{2}, +\infty)$$

$$\therefore R(f) = (-\infty, -\tfrac{\sqrt{2}}{2}] \cup [\tfrac{\sqrt{2}}{2}, +\infty)$$

$$f(x) = \sqrt{2 - |\ln(1-x)|}$$

$$D(f) = \{x \mid 2 - |\ln(1-x)| \geq 0\} \cap \{x \mid 1-x > 0\}$$
$$= \{x \mid -2 \leq \ln(1-x) \leq 2\} \cap \{x \mid x < 1\}$$
$$= \{x \mid 1-e^{2} \leq x \leq 1-e^{-2}\} \cap \{x \mid x < 1\}$$
$$= [1-e^{2}, 1-e^{-2}]$$

$R(f) = [0, \sqrt{2}]$ for $0 \leq 2 - |\ln(1-x)| \leq 2$

Remark: domains and ranges of some important functions:

1). $D(\frac{1}{x}) = (-\infty, 0) \cup (0, +\infty) = \{x \mid x \neq 0\}$, $R(f) = (-\infty, 0) \cup (0, +\infty)$

2) $f(x) = \cos x$. $D(f) = R = (-\infty, +\infty)$, $R(f) = [-1, 1]$

3) $f(x) = \sqrt{x}$ $D(f) = [0, +\infty) = \{x \mid x \geq 0\}$. $R(f) = [0, +\infty)$

(4) $f(x) = \ln x$ $D(f) = (0, +\infty)$, $R(f) = (0, +\infty)$

2. Graph of functions.

1. $f(x) = ||x-2|-4|$

Step by step

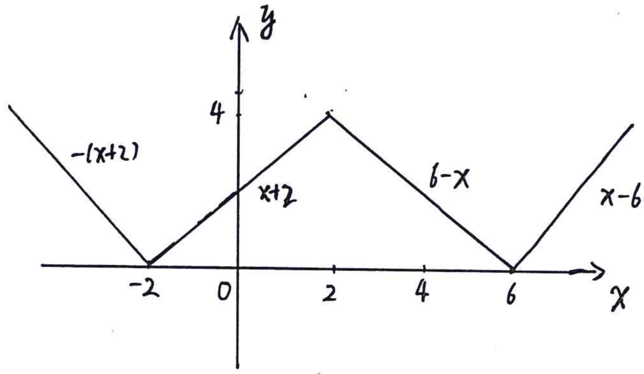
$|x-2|$

$x \geq 2$: $f(x) = |x-2-4| = |x-6|$

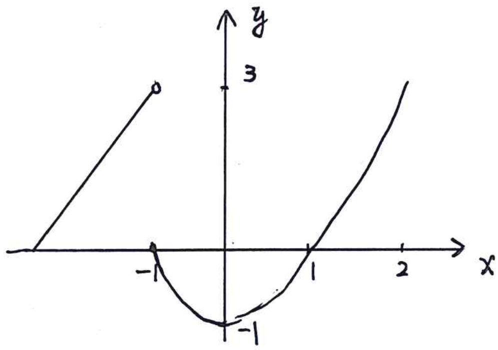
- $x \geq 6$: $f(x) = x-6$
- $2 \leq x < 6$: $f(x) = 6-x$

$x < 2$: $f(x) = |2-x-4| = |-x-2| = |x+2|$

- $-2 \leq x < 2$: $f(x) = x+2$
- $x < -2$: $f(x) = -(x+2)$



2. $f(x) = \begin{cases} 2x+5 &, x<-1 \\ x^2-1 &, x \geq -1 \end{cases}$



MATH 1010 tutorial 1

prepared by Chung Shun Wai

Topics : Domain & Range

Question : Find the maximum domain of definition and range with that domain of the function $f(x)$?

1. $f(x) = \sqrt{7-2x}$; 2. $f(x) = \dfrac{1}{\sin x}$

3. $f(x) = \dfrac{2}{1-\ln(x)}$; 4. $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

Q1. Given $f(x) = \sqrt{7-2x}$

- $\text{Domain}(f) = \{x \in \mathbb{R} \mid 7 - 2x \geq 0\}$

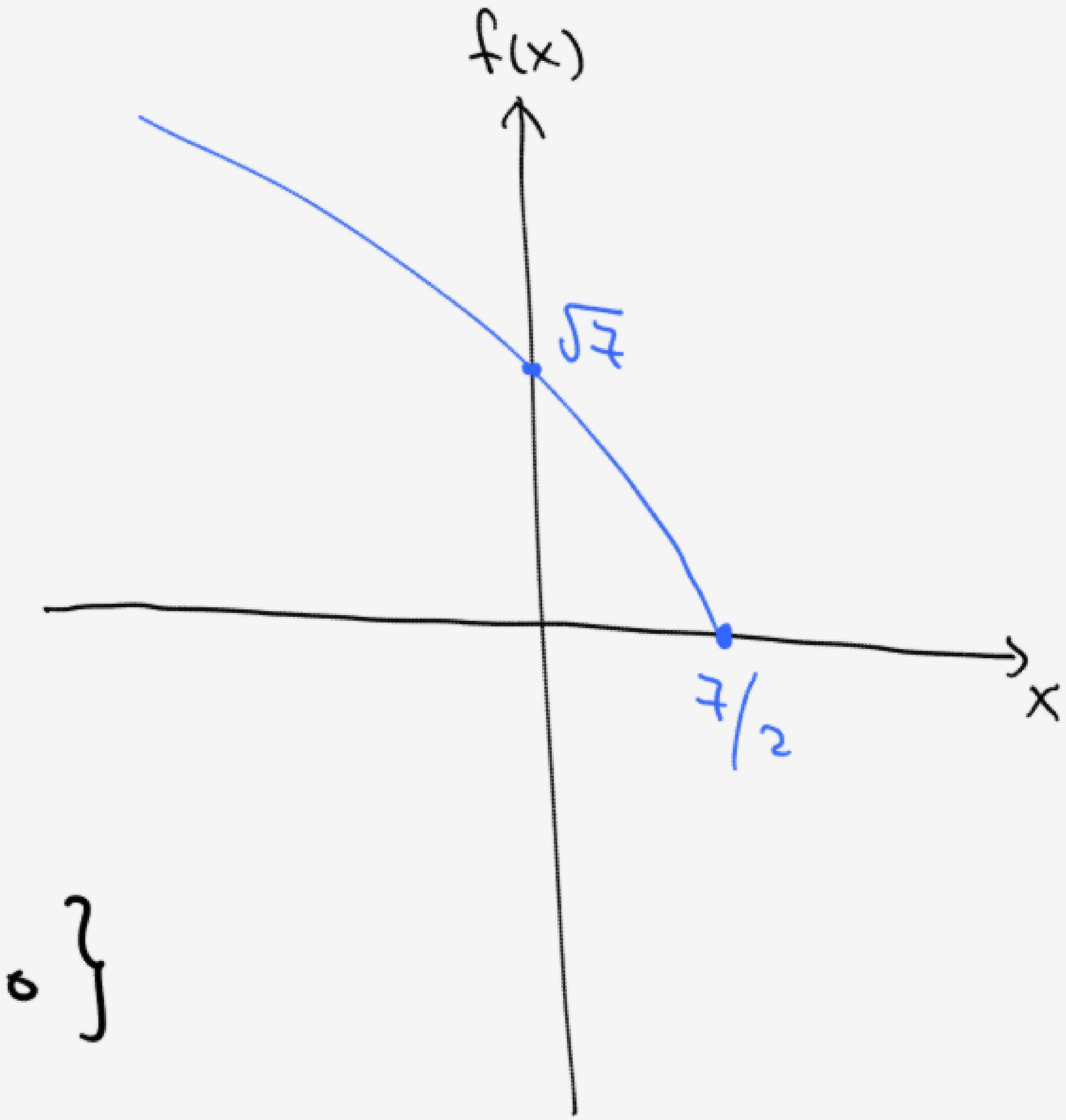
$= \{x \in \mathbb{R} \mid x \leq 7/2\}$

$= (-\infty, 7/2]$

- $\text{Range}(f) = \{f(x) \in \mathbb{R} \mid x \in \text{Domain}(f)\}$

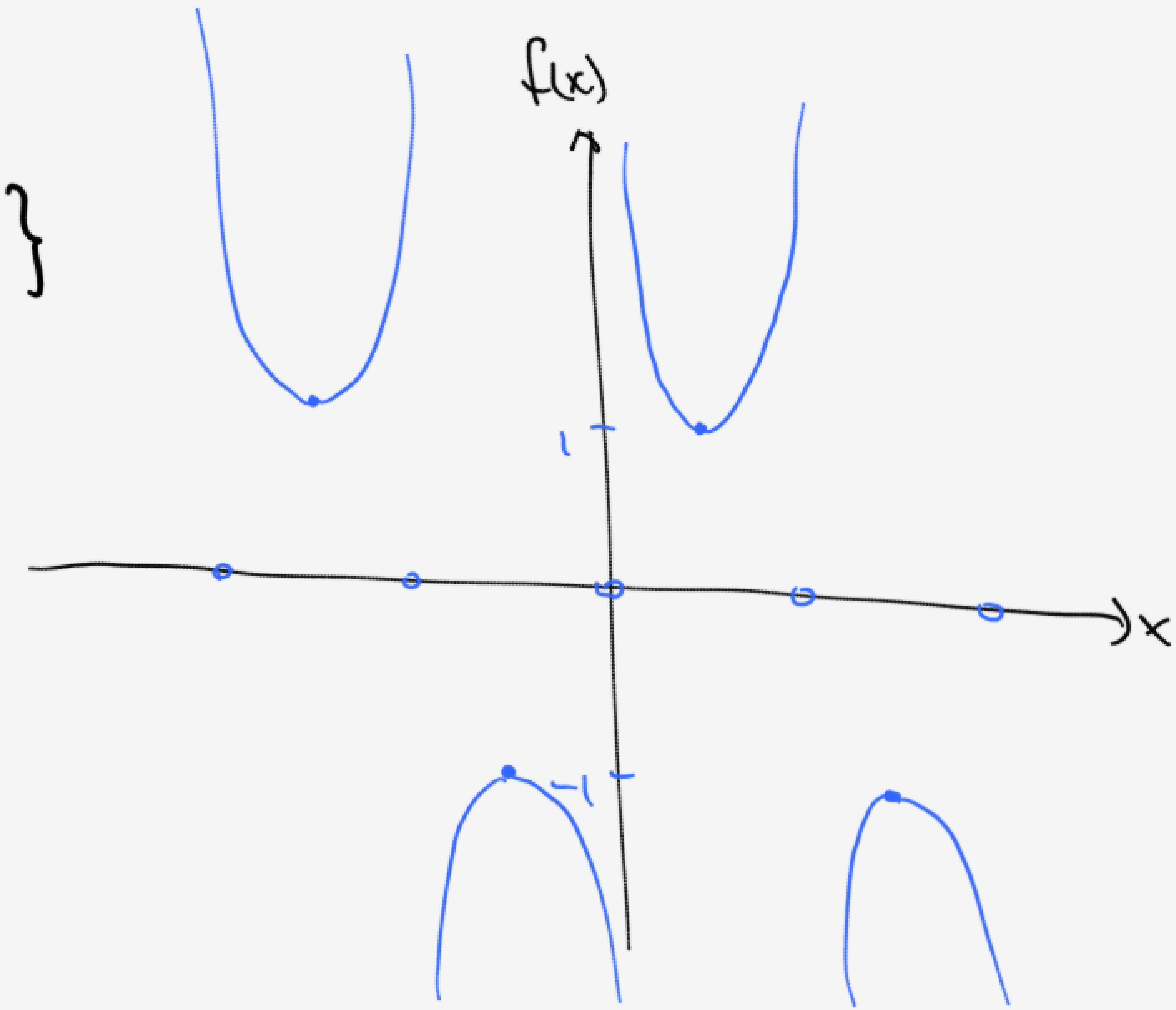
$= \{f(x) = \mathbb{R} \mid f(x) = \sqrt{7-2x} \geq 0\}$

$= [0, \infty)$



Q2. Given $f(x) = \frac{1}{\sin x}$,

- $\text{Domain}(f) = \{x \in \mathbb{R} \mid \sin x \neq 0\}$
$= \{x \in \mathbb{R} \mid x \neq k\pi, k \in \mathbb{Z}\}$
$= \mathbb{R} \setminus \mathbb{Z}\pi$



- $\text{Range}(f)$:

$\forall x \in \text{Domain}(f) = \mathbb{R} \setminus \mathbb{Z}\pi$,
$\text{Range}(\sin x) = [-1, 0) \cup (0, 1]$

$\Rightarrow \text{Range}(f) = \text{Range}\left(\frac{1}{\sin x}\right)$
$= (-\infty, -1] \cup [1, \infty) = \mathbb{R} \setminus (-1, 1)$

Q3. $f(x) = \dfrac{2}{1-\ln(x)}$

- $\text{Domain}(f) = \{x \in \mathbb{R} \mid 1-\ln(x) \neq 0 \ \& \ x > 0\}$

$= \{x \in \mathbb{R} \mid x \neq e \ \& \ x > 0\}$

$= \mathbb{R}^+ \setminus \{e\}$

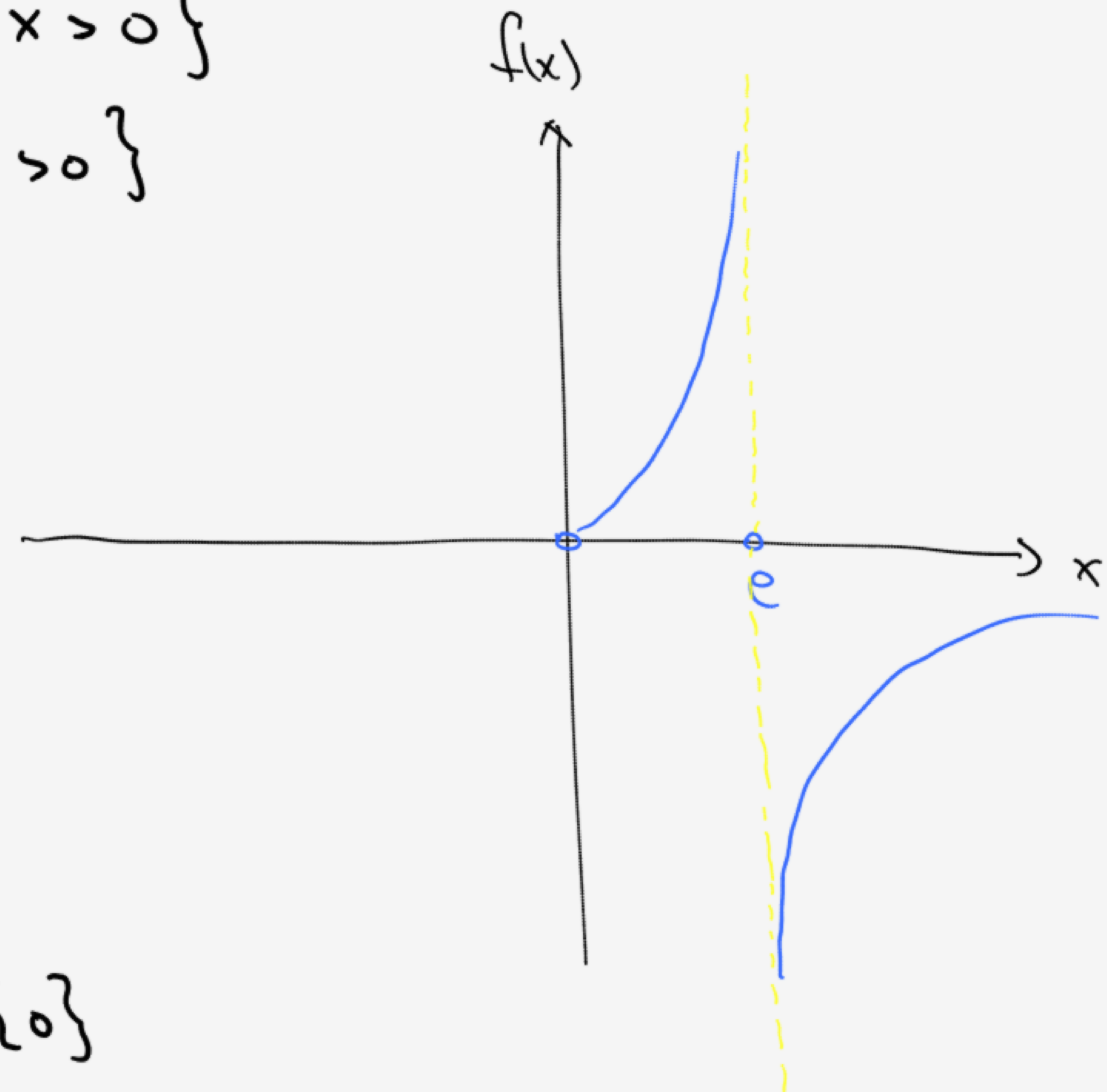
- Range(f):

$\forall x \in \text{Domain}(f) = \mathbb{R}^+ \setminus \{e\}$

$\text{Range}(\ln x) = \mathbb{R} \setminus \{1\}$

$\Rightarrow \text{Range}(1-\ln x) = \mathbb{R} \setminus \{0\}$

$\Rightarrow \text{Range}(f) = \text{Range}\left(\dfrac{2}{1-\ln x}\right) = \mathbb{R} \setminus \{0\}$



Q4. Given $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

- Note that for any values of $x \in \mathbb{R}$, $f(x)$ is well-defined. Hence, $\text{Domain}(f) = \mathbb{R}$.

- <u>Range(f):</u>

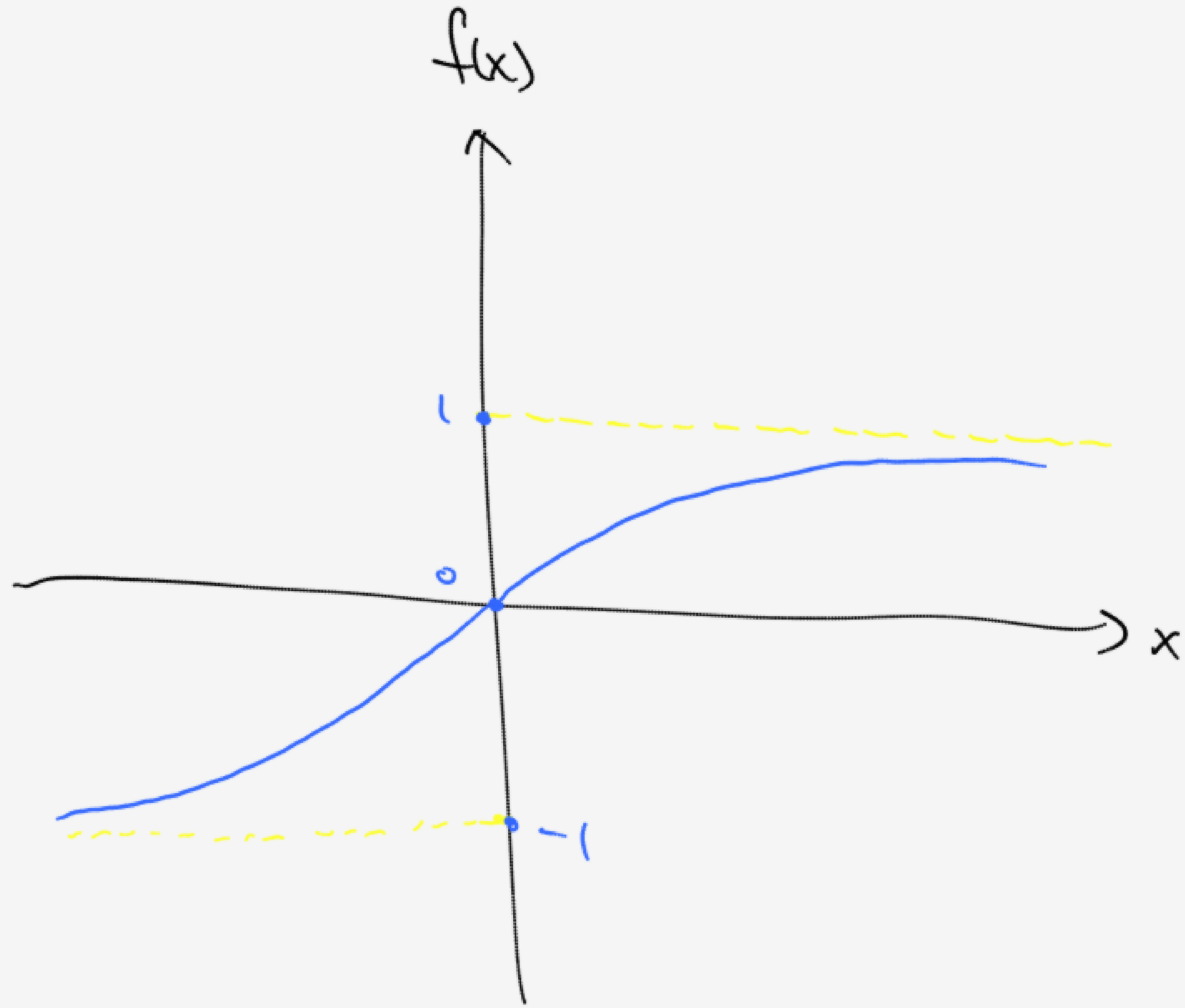
① claim: $\text{Range}(f) \subseteq (-1,1)$

Note that $\forall x \in \mathbb{R}$, $x^2 < x^2+1$

$\Rightarrow \quad |x| < \sqrt{x^2+1}$

$\Rightarrow \quad |f(x)| = \left|\dfrac{x}{\sqrt{x^2+1}}\right| < 1$

$\Rightarrow \quad \text{Range}(f) \subseteq (-1,1)$



② claim: $(-1,1) \subseteq \text{Range}(f)$.

Fix a $y \in (-1,1)$

by solving $f(x) = y$ for x,

we have $y = \dfrac{x}{\sqrt{x^2+1}}$ — (*)

$$\Rightarrow \quad y^2 = \frac{x^2}{x^2+1}$$

$$\Rightarrow \quad \frac{1}{y^2} = 1 + \frac{1}{x^2}$$

$$\Rightarrow \quad x^2 = \frac{y^2}{1-y^2}$$

$$\Rightarrow \quad x = \frac{\pm y}{\sqrt{1-y^2}}$$

notice that eqt (*) suggests that x, y must have same sign.

so $$x = \frac{+y}{\sqrt{1-y^2}}$$

hence we have that

$\forall y \in (-1,1)$, $\exists x \in \mathbb{R}$ s.t.

$$y = f(x) \in \text{Range}(f)$$

Hence $(-1,1) \subseteq \text{Range}(f)$

- Combining ①, ② we thus have $\text{Range}(f) = (-1,1)$.

11/9/2014 Math 1010C, Tutorials

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Content: Chapter 1, functions:

- def'n of f'ns; max. domain of def'n; graphs of f'ns; injectivity, surjectivity, the range of a f'n etc.
- 11/9: algebra & composition of f'ns, & properties of some elementary f'ns;

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1. Find the maximum domain of def'n of the following f'ns, and the range of the function with this domain.

(a) $f(x) = x^3 - 3x + 5$;

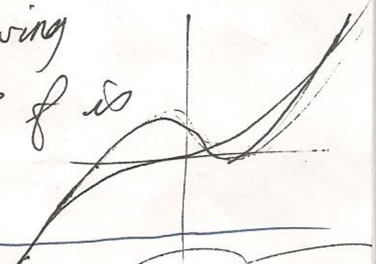
(c). $f(x) = \dfrac{x+4}{x^2-3x-10}$; ? $f(\mathbb{R}) = \mathbb{R} \setminus \left(\dfrac{1}{-11+6\sqrt{2}}, \dfrac{1}{11+6\sqrt{2}}\right)$

(k) $f(x) = \ln(\ln x)$;

(l) $f(x) = \sqrt{2 - |\ln(1-x)|}$;

Solutions: (a) $x \in \mathbb{R}$; & $\underline{f(\mathbb{R}) = \mathbb{R}}$;

you can persuade yourself by showing $f(-\infty) = -\infty$, $f(+\infty) = +\infty$, & f is "continuous".



See next page. (c)

~~(c)~~ $x \in \mathbb{R} \setminus \{-2, 5\} = (-\infty, -2) \cup (-2, 5) \cup (5, +\infty)$.

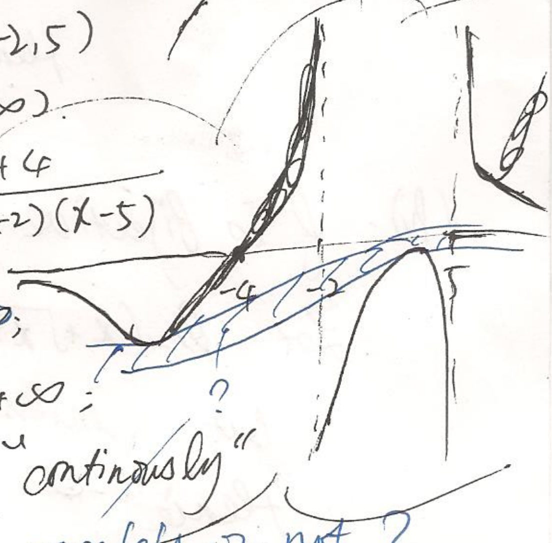
Answer ~~$f(\mathbb{R}) = \mathbb{R}$~~

$f(\mathbb{R}) = \mathbb{R} \setminus \boxed{\left(\dfrac{1}{-11+6\sqrt{2}}, \dfrac{1}{11+6\sqrt{2}}\right)}$?

- $f(x) = \dfrac{x+4}{(x+2)(x-5)}$
- $f(-\infty) = \cancel{-\infty}\ 0$;

$f(-2-\varepsilon) = +\infty$;

$f(-4) = 0$; "continuously"



?

overlaps or not?

(k). $f(x) = \ln(\underbrace{\ln x})$

$$\ln x > 0 \iff x > 1;$$

max domain of def'n is $(1, +\infty)$, & $f((1,+\infty)) = \mathbb{R}$;

(l). $f(x) = \sqrt{\underbrace{2 - |\ln(1-x)|}}$.

$$2 - |\ln(1-x)| \geq 0 \iff |\ln(1-x)| \leq 2$$

$$\iff -2 \leq \ln(1-x) \leq 2 \iff e^{-2} \leq 1-x \leq e^{2}$$

$$\iff 1-e^{2} \leq x \leq 1-e^{-2};$$ & $f([1-e^{2}, 1-e^{-2}]) = [0, \sqrt{2}]$;

$\square$

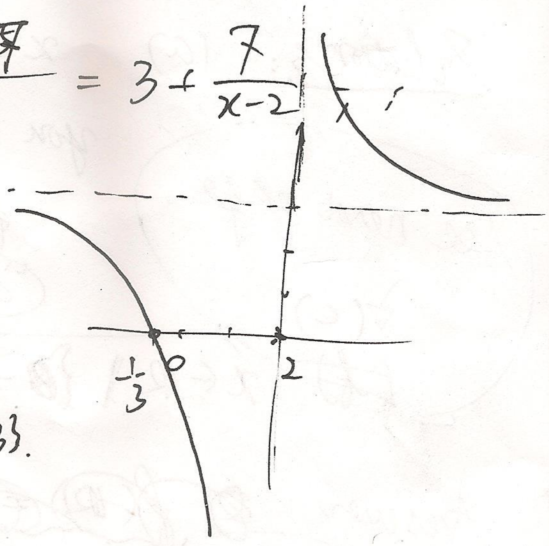
2. For each of the following fns, determine whether it is injective, surjective or bijective.

(e). $f: \mathbb{R} \setminus \{2\} \to \mathbb{R}$; $f(x) = \dfrac{3x+1}{x-2}$;

(h): $f: \mathbb{R} \to \mathbb{R}$; $f(x) = \ln(x + \sqrt{x^2+1})$;

<u>Solutions</u>: (e). Injective, & $f(\mathbb{R} \setminus \{2\}) = \mathbb{R} \setminus \{3\}$.

Reason $f(x) = \dfrac{3x-6+7}{x-2} = 3 + \dfrac{7}{x-2}$;



since $g(x) = \frac{7}{x}$ is injective from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R} \setminus \{0\}$.

Hence $f(x)$ is injective from $\mathbb{R} \setminus \{2\}$ to $\mathbb{R} \setminus \{3\}$.

(h): f is bijective from $\mathbb{R}$ to $\mathbb{R}$:

Let $\ln(x + \sqrt{x^2+1}) = y$, then $x + \sqrt{x^2+1} = e^{y}$; ①

But inverse of ① gives: $-x + \sqrt{x^2+1} = e^{-y}$; ②

Hence ①−②: $x = \frac{1}{2}(e^{y} - e^{-y})$. & clearly max domain of def'n ~~can be~~ is $\mathbb{R}$.

$\square$

11/9/2014

1 (a) $f(x) = \dfrac{x+4}{x^2-3x-10}$;

maximal domain is $\mathbb{R} \setminus \{-2, 5\}$;

for the range, • f can achieve 0: $x = -4$;

• Can f achieve $a \neq 0$?

i.e. $\exists ?\ x \in \mathbb{R}\setminus\{-2,5\}$, s.t. $\dfrac{x+4}{x^2-3x-10} = a$.

$$\iff x^2 - 3x - 10 = \frac{x}{a} + \frac{4}{a}$$

$$\iff x^2 - \left(3 + \frac{1}{a}\right)x - 10 - \frac{4}{a} = 0$$

this has solution iff

$$\Delta = \left(3+\frac{1}{a}\right)^2 - 4\left(-10 - \frac{4}{a}\right)$$

$$= \frac{1}{a^2} + \frac{22}{a} + 49$$

$$= \left(\frac{1}{a} + 11\right)^2 + \underbrace{49 - 121}_{(-72)} \geq 0.$$

$$\iff \left|\frac{1}{a} + 11\right| \geq \sqrt{72} = 6\sqrt{2};$$

$$\iff \frac{1}{a} + 11 \geq 6\sqrt{2} \text{ or } \frac{1}{a} + 11 \leq -6\sqrt{2};$$

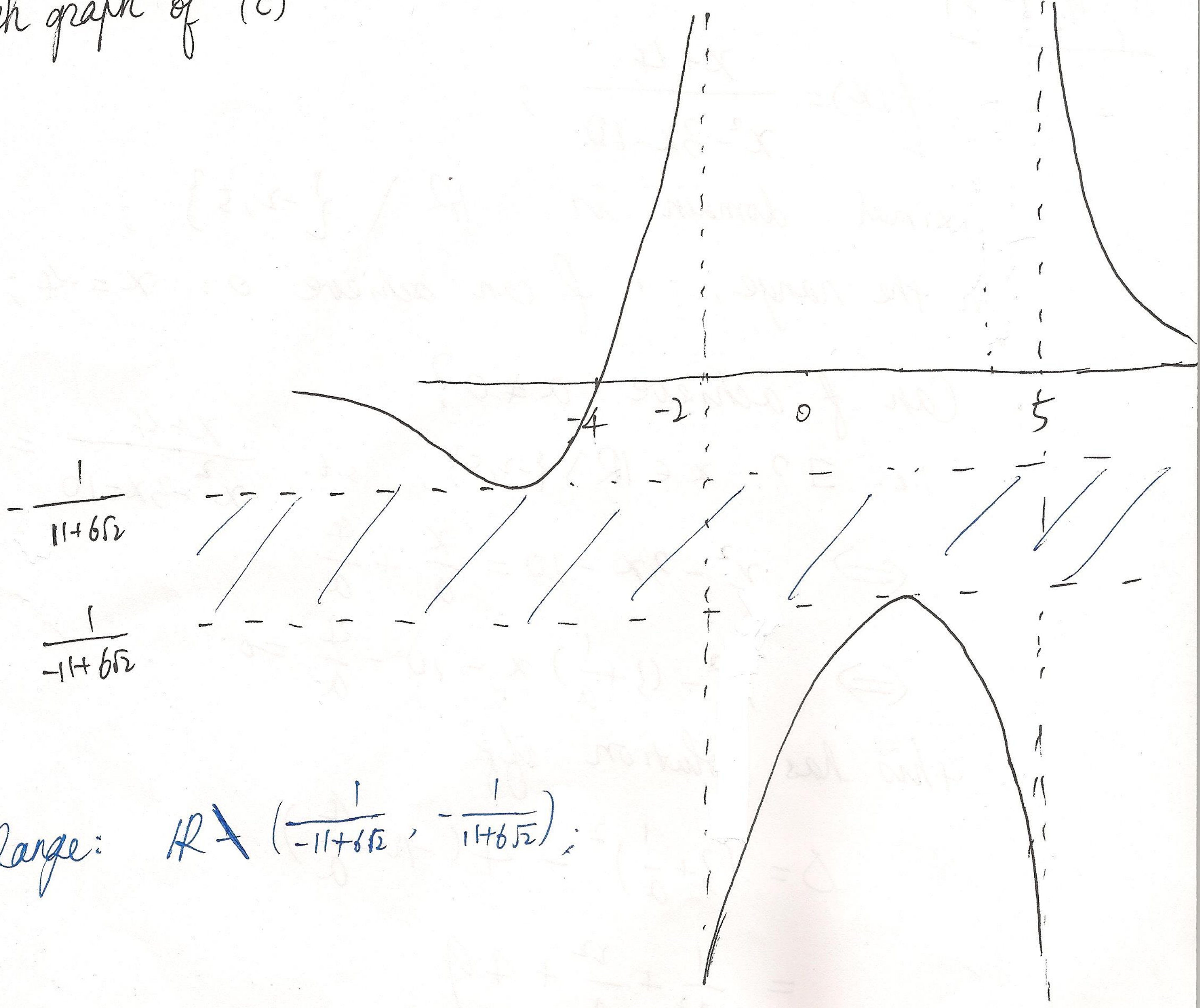
$$\iff a \in \left(-\infty, \frac{1}{-11+6\sqrt{2}}\right] \cup \left[-\frac{1}{11+6\sqrt{2}}, 0\right) \cup (0, +\infty)$$

Hence the range is

$$f(\mathbb{R}\setminus\{-2,5\}) = \mathbb{R} \setminus \left(\underset{-\frac{11+6\sqrt{2}}{49}}{\frac{1}{-11+6\sqrt{2}}}, \underset{\frac{-11+6\sqrt{2}}{49}}{-\frac{1}{11+6\sqrt{2}}}\right) \quad \square$$

P1 +1

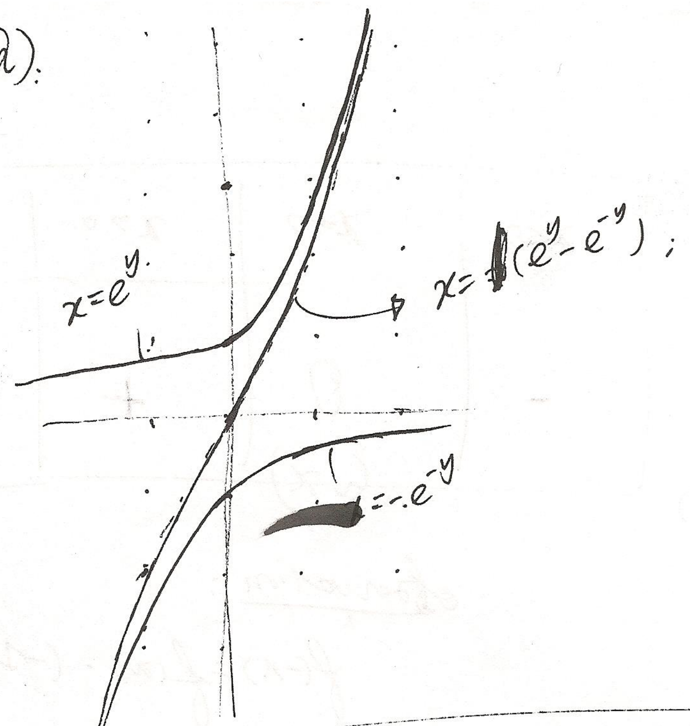
Rough graph of (c)

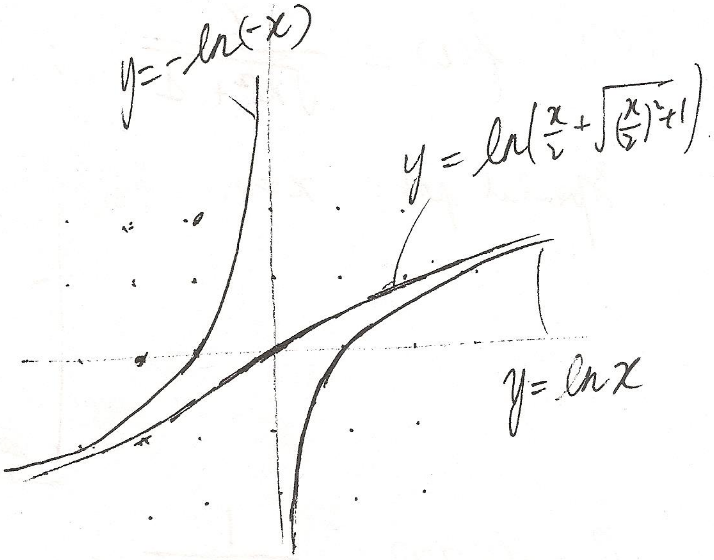


Range: $\mathbb{R} \setminus \left(\frac{1}{-11+6\sqrt{2}}, -\frac{1}{11+6\sqrt{2}}\right)$;

$\square$ of 1(c).

(2h).





3. Sketch the graph of the following f'ns:

(c) $f(x)=\dfrac{x^2}{x-2}$, $x\neq 2$;

(e) $f(x)=\dfrac{x}{\sqrt{x^2+1}}$, $x\in\mathbb{R}$;

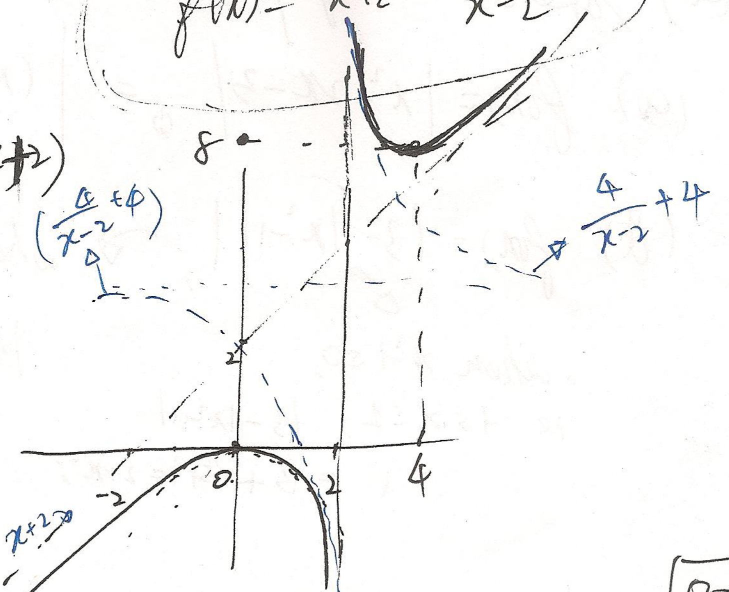
Solutions:

(c). Guiding principles:
I. Find special pts & special values;
II. Transform, express the f'ns in terms of f'ns that we are familiar with;

Special pts:

$-\infty$	$x<0$	$x=0$	$0<x<2$	$x=2$	$x>2$	$+\infty$
$-\infty$ $(\sim x+2)$	$-$	0 $(\sim -\frac{x^2}{2})$	$-$	∞ $(\sim\frac{4}{x-2})$	$+$	$+\infty$ $(\sim x+2)$

$f(x)=x+2+\dfrac{4}{x-2}$



(e). $f(x) = \dfrac{x}{\sqrt{x^2+1}}$;

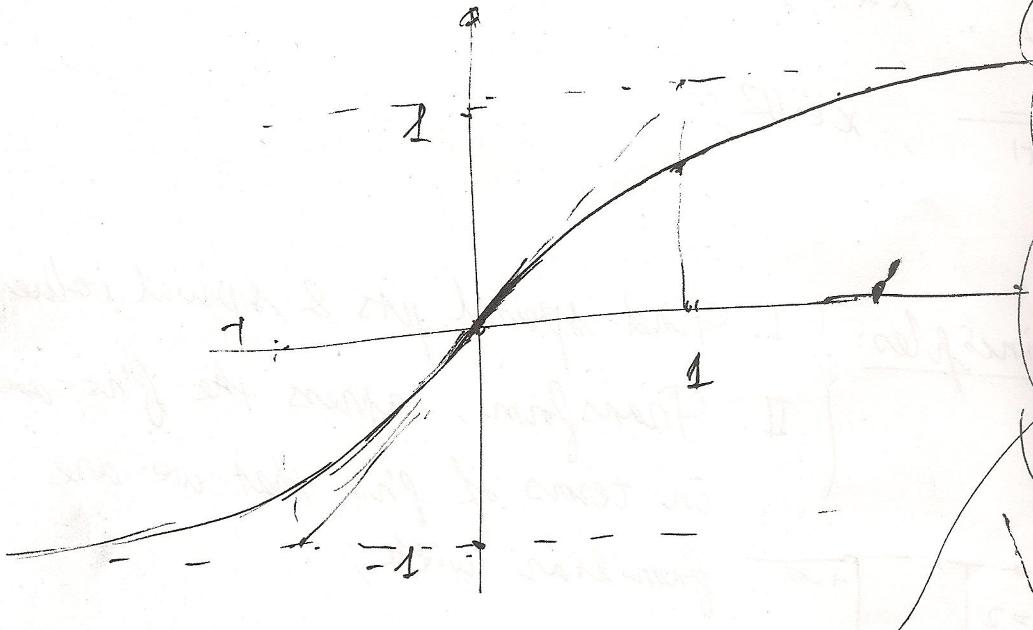
Special pts: $x=0$.

$-\infty$	$x<0$	$x=0$	$x>0$	$+\infty$
-1	$-$	0	$+$	$+1$
$\sim(-1)$		$(\sim x)$		$\sim(+1)$

$$f(x) = \begin{cases} x>0, & \dfrac{1}{\sqrt{1+\frac{1}{x^2}}} ; \\ x<0, & -\dfrac{1}{\sqrt{1+\frac{1}{x^2}}} ; \end{cases}$$

observation:

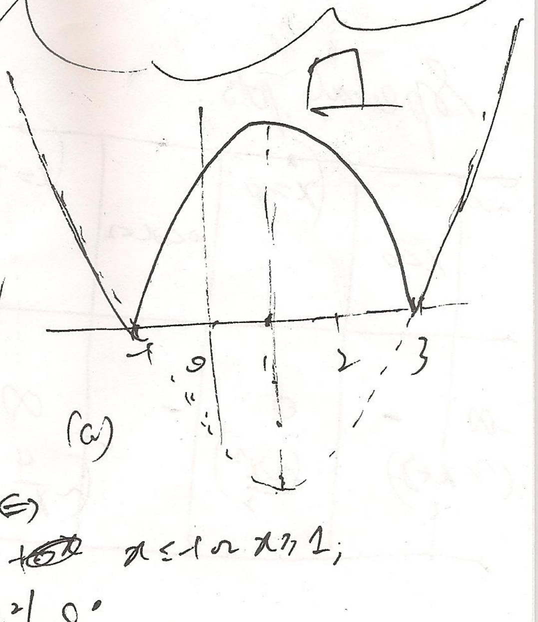
$f(-x) = f(x)\cdot(-1)$



Principle:
Classify / Enumerate all situations & analyze separately;

(4). Sketch the graphs

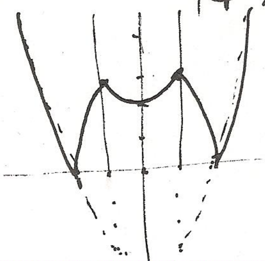
(c). $f(x) = |x^2 - 2x - 3| = |(x-3)(x+1)|$.



(d) $f(x) = |3 - |x^2-1||$; • when $x^2-1 \geq 0$, $\Leftrightarrow$ $x \leq -1$ or $x \geq 1$;

$|3 - x^2 + 1| = |4 - x^2|$;

• when $x^2-1 \leq 0$,
i.e. $-1 \leq x \leq 1$, $|3 - |x^2-1||$
$= |3 + x^2 - 1| = 2 + x^2$;



$\square$

Tutorial 1 (MATH1010. B.F) Min, Jie jmin@math.cuhk.edu.hk

便笺标题 2014/9/11

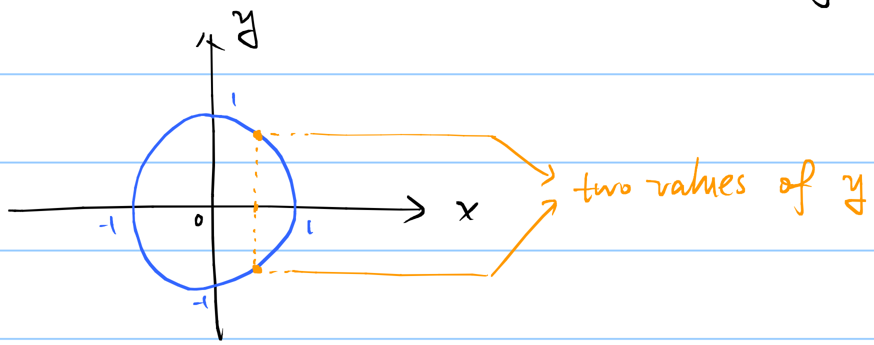
* <u>Functions</u>. determine whether y is a function of x

("function" if for every x in domain, there is a unique y)

(1) $x^2+y^2=1$, $-1\le x\le 1$ is <u>NOT</u> a function.

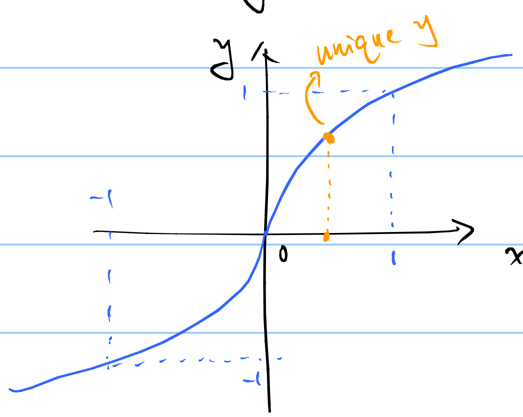
Because $\Leftrightarrow y^2=1-x^2 \Leftrightarrow y=\pm\sqrt{1-x^2}$

for each $-1<x<1$, there two values of y



(2) $y^3=x$, $-1\le x\le 1$ is a function

Because $y=\sqrt[3]{x}$ for each x there is a unique y



* Maximal domain of a function

(1) $f(x) = \dfrac{1}{x-1}$

In order that $\dfrac{1}{x-1}$ is defined, we need $x-1 \neq 0$

that is $x \neq 1$, then the maximal domain

$$D = \{x \in \mathbb{R} \mid x \neq 1\} = (-\infty, 1) \cup (1, +\infty)$$

you may write it in either of the two forms

(2) $f(x) = \dfrac{1}{x^2 - 6x + 2}$

In order that f is defined, we need $x^2 - 6x + 2 \neq 0$

Find sol'n to $x^2 - 6x + 2 = 0$ use the formula $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x_1 = \frac{6 + \sqrt{36-8}}{2} = 3 + \sqrt{7}$$

$$x_2 = \frac{6 - \sqrt{36-8}}{2} = 3 - \sqrt{7}$$

So the maximal domain for f is

$$D = \{x \in \mathbb{R} \mid x \neq 3+\sqrt{7},\ x \neq 3-\sqrt{7}\}$$
$$= (-\infty, 3-\sqrt{7}) \cup (3-\sqrt{7}, 3+\sqrt{7}) \cup (3+\sqrt{7}, +\infty)$$

(3) $f(x) = \dfrac{1}{\sqrt{x^2 - 6x + 5}}$

In order that f is defined, we need

$$\begin{cases} \sqrt{x^2 - 6x + 5} \neq 0 \\ x^2 - 6x + 5 \geq 0 \end{cases} \iff x^2 - 6x + 5 > 0$$

Since $x^2 - 6x + 5 = (x-1)(x-5)$

then $x^2-6x+5>0 \iff (x-1)(x-5)>0$

$\iff x>5$ or $x<1$

$$D = \{x \in \mathbb{R} \mid x<1 \text{ or } x>5\}$$

$$= (-\infty, 1) \cup (5, +\infty)$$